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On the applicability of the layered sine–Gordon model for Josephson-coupled high-T_c layered superconductors

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Abstract
We find a mapping of the layered sine–Gordon model to an equivalent gas of topological excitations and determine the long-range interaction potentials of the topological defects. This enables us to make a detailed comparison to the so-called layered vortex gas, which can be obtained from the layered Ginzburg–Landau model. The layered sine–Gordon model has been proposed in the literature as a candidate field-theoretical model for Josephson-coupled high-T_c superconductors, and the implications of our analysis for the applicability of the layered sine–Gordon model to high-T_c superconductors are discussed. We are led to the conjecture that the layered sine–Gordon and the layered vortex gas models belong to different universality classes. The determination of the critical temperature of the layered sine–Gordon model is based on a renormalization-group analysis.

1. Introduction

Two essential prerequisites for an analysis of superconducting materials are anisotropic models, as initiated by Ginzburg [1], and the inclusion of vortices, as envisaged by Abrikosov [2]. Typical high-transition-temperature superconductors consist of copper oxide superconducting planes separated by insulating layers. In the phenomenological description of high-T_c superconductivity, one may use an anisotropic, continuous Ginzburg–Landau theory [1, 3, 4], but only for not too large anisotropy. Note that the anisotropic, continuous model can be mapped onto the isotropic Ginzburg–Landau model by an appropriate rescaling method [5]. However, in the case of extremely high anisotropy like in $Bi_2Sr_2CaCu_2O_8$, the discreteness of the structure becomes relevant [6], and it becomes necessary to use a layered Ginzburg–Landau model [3, 7] where the layers are coupled by Josephson or electromagnetic interactions.

This provides a good basis for the discussion of the vortex-dominated properties of high-T_c superconductors. Some exact and some approximate mappings of the layered Ginzburg–Landau model (i.e. the Lawrence–Doniach model [7]) onto various other statistical, field-theoretical or spin models, like the layered vortex gas [3, 8–12] and the anisotropic XY models [13–15], have already been discussed in the literature, and these models have also been proposed and used for the description of the vortex dynamics in high-T_c superconductors. Connections to sine–Gordon-type models [16–23] have also been explored in the literature.

The latter investigations are motivated by the well-known fact that the massless two-dimensional (2D) sine–Gordon scalar field theory belongs to the universality class of the 2D–XY spin model and consequently to that of the 2D Coulomb or vortex gas. The mappings between these models and also the phase structure have been discussed in the literature in great detail (see, e.g., [8–12, 16–18, 24–29]). Since the layered Ginzburg–Landau model can be considered as the continuum limit of the anisotropic 3D–XY model (discrete in the z-direction but continuous in the xy-planes), one might suggest that the field-theoretical counterpart of the layered Ginzburg–Landau model should be a sine–Gordon-type model. However, the 3D Ginzburg–Landau theory (in the London limit and in the absence of electromagnetic fields), which can be considered as the continuum limit of the 3D–XY planar rotator, and the 3D sine–Gordon model do not belong to the same universality class (see [30–32]), a phase transition being absent in the 3D sine–Gordon case. Since layered models are always constructed from 3D models by a suitable discretization of the derivative term in one of the spatial dimensions (see, e.g., [33]), the equivalence of the layered vortex gas and layered sine–Gordon models remains questionable. One purpose of this paper is to clarify this point by finding an exact mapping of the layered sine–Gordon model to an equivalent gas of topological excitations, which in turn can be compared directly to the layered vortex gas. We also consider the phase structure and the critical behaviour of the N-layer sine–Gordon model by a renormalization group method, and determine the relation of the critical parameter b_c^2 of the layered sine–Gordon model to the critical temperature, as a function of the number of layers.

This paper is organized as follows. In section 2, we discuss the comparison of the layered Ginzburg–Landau model to the layered sine–Gordon model, by a mapping of each model to an equivalent gas of topological excitations. In section 3, we discuss the renormalization-group (RG) flow of the layered sine–Gordon model. Conclusions are reserved for section 4.

2. Layered Ginzburg–Landau model versus layered sine–Gordon model

2.1. Mapping of the layered Ginzburg–Landau model to the layered vortex gas

The Ginzburg–Landau theory has been developed by applying a variational method to an assumed expansion of the free energy in powers of $|\psi|^2$ and $|\partial_\mu \psi|^2$, where ψ is a complex order parameter (the inhomogeneous condensate of the superconducting electron pairs) and $|\psi|^2$ represents the local density of superconducting electron pairs (for a detailed discussion see, e.g., [34]). Its detailed form can be found in equations (6-6) and (6-9) of [35]. Upon a discretization of one of the spatial directions (say, the z-coordinate), one obtains the layered Ginzburg–Landau (or Lawrence–Doniach [7]) model with the free energy (in natural units: $\hbar = c = \epsilon_0 = 1$),

$$F = s \int \mathrm{d}^2 r \left(\sum_{n=1}^{N} \left(\alpha |\psi_n|^2 + \frac{\beta}{2} |\psi_n|^4 + \frac{|\partial_x \psi_n|^2 + |\partial_y \psi_n|^2}{2\, m_{ab}} \right) + \sum_{n=1}^{N-1} \frac{|\psi_{n+1} - \psi_n|^2}{2\, m_c\, s^2} \right). \tag{1}$$

Here, m_{ab} and m_c represent the intralayer and interlayer effective masses, s is the interlayer distance, and N stands for the total number of layers. The sum over μ covers the spatial coordinates $\mu = x, y, z$. The parameters α and β are discussed in equations (6-6) and (6-8) of [35]. In order to investigate the vortex dynamics in the framework of the Ginzburg–Landau theory, one has to consider the discretized model given in equation (1) in the London approximation. One writes the complex, layer-dependent order parameter as $\psi_n(\vec{r}) = \psi_{0,n}(\vec{r})\exp[\mathrm{i}\phi_n(\vec{r})]$ with real $\psi_{0,n}(r)$, where the $\phi_n \in [0, 2\pi)$ are compact variables, and the moduli $\psi_{0,n}$ are assumed to be constant and identical in every layer (i.e. $\psi_{0,n}(\vec{r}) = \psi_0$), which is the London-type approximation. The London-type form of the layered Ginzburg–Landau model with Josephson coupling can be mapped (see [3, 8–12]) onto the layered vortex gas. The globally neutral layered vortex gas with N layers is characterized by the partition function (see equation (2.3) of [11])

$$\mathcal{Z}_{\mathrm{LVG}} = \sum_{\nu=0}^{\infty} \frac{z^{2\nu}}{(\nu!)^2} \sum_{n_1=1}^{N} \int \frac{\mathrm{d}^2 r_1}{a^2} \cdots \sum_{n_{2\nu}=1}^{N} \int \frac{\mathrm{d}^2 r_{2\nu}}{a^2} \times \sum_{\substack{\sigma_1,\ldots,\sigma_\nu=\pm1 \\ \sigma_{\nu+\gamma}=-\sigma_\gamma,\ \gamma\in\{1,\ldots\nu\}}} \exp\left(-\frac{1}{k_{\mathrm{B}}T} \sum_{\alpha\neq\beta} \frac{1}{2} \sigma_\alpha\, \sigma_\beta\, V(r_{\alpha\beta},\, n_{\alpha\beta})\right), \tag{2}$$

where $\sigma_\alpha = \pm 1$ is the charge of the αth vortex, a stands for the lattice spacing, k_{B} is the Boltzmann constant, T is the temperature, and the interaction potential V between two vortices depends on their relative distance $r_{\alpha\beta}$ within the two-dimensional planes ($r_{\alpha\beta} = |\vec{r}_\alpha - \vec{r}_\beta|$) and on the distance $n_{\alpha\beta}$ across the planes ($n_{\alpha\beta} = |n_\alpha - n_\beta|$), where n_α is the layer in which the αth vortex is located. There are 2ν vortices with fugacity z and these fulfil the neutrality condition $\sum_{\alpha=1}^{2\nu} \sigma_\alpha = 0$. The positive (negative) vorticity is represented by positive (negative) charges. The restriction to globally neutral charge configurations is ensured by the condition $\sigma_{\nu+\gamma} = -\sigma_\gamma$ for $\gamma \in \{1, \ldots \nu\}$ in equation (2). Following [3, 8–12], we neglect interactions between vortices separated by more than one layer, and this results in intralayer and interlayer interaction potentials which have commonly accepted short-range and long-range asymptotic forms given by (see, e.g., equations (2.5) and (2.6) of [11])

$$V(r_{\alpha\beta}, n_{\alpha\beta} = 0) = -\ln\left(\frac{r_{\alpha\beta}}{a}\right) - \sqrt{\lambda}\frac{r_{\alpha\beta} - a}{a}, \tag{3a}$$

$$V(r_{\alpha\beta}, n_{\alpha\beta} = 1) = b\sqrt{\lambda}\frac{r_{\alpha\beta}}{a}. \tag{3b}$$

The coupling $\lambda \sim a^2 J_\perp / J_\parallel$ is proportional to the ratio of the interlayer Josephson coupling $J_\perp$ to the intralayer coupling $J_\parallel$, and b is a constant of order unity. The intralayer interaction between the vortices is logarithmic for short distances, as in the case of the usual 2D Coulomb or vortex gas, but linear for large distances. The interlayer interaction is always linear and similar to the long-range intralayer interaction, but with an opposite sign. Within a layer, vortices of opposite charge attract, whereas the positive prefactor of the linear term in the interlayer interaction implies the formation of vortex stacks of like charges.

2.2. Mapping of the layered sine–Gordon model to an equivalent gas of topological excitations

The well-known sine–Gordon model in Euclidean space is defined via the action

$$S_{\mathrm{SG}}[\varphi] = \int \mathrm{d}^2 r \left(\tfrac{1}{2}(\partial_\mu \varphi)^2 - y\cos(b\,\varphi)\right), \tag{4}$$

where the minus sign of the periodic term is chosen so that the zero-field configuration remains a (local, not global, infinitely degenerate) minimum. As usual, φ here is a dimensionless scalar field, $(\partial_\mu\varphi)^2 \equiv \sum_{\mu=1}^{2} (\partial_\mu\varphi)^2$, y is a fundamental Fourier amplitude, and b is a dimensionless frequency. This model is well known to describe the Kosterlitz–Thouless–Berezinskii (KTB) phase transition [36] in two dimensions. If one adds a second layer which leads to the appearance of two fields φ_1 and φ_2, one may devise the following natural ansatz for the interlayer interaction term, $\frac{1}{2}J(\varphi_1 - \varphi_2)^2$, where J is the Josephson-type coupling whose physical dimension is equal to the square of the inverse length. Indeed, the layered sine–Gordon model with this particular interlayer interaction term has been proposed in [16, 17] for the description of the vortex properties of Josephson-coupled layered superconductors. The double-layer sine–Gordon model [16–18, 20, 37, 38] is thus given by the Euclidean action

$$\begin{aligned} S_{\mathrm{2LSG}} &= \int \mathrm{d}^2 r \left(\tfrac{1}{2}\sum_{n=1}^{2} (\partial_\mu \varphi_n)^2 + \tfrac{1}{2} J(\varphi_1 - \varphi_2)^2 - y \sum_{n=1}^{2} \cos(b\varphi_n) \right) \\ &= \int \mathrm{d}^2 r \left(\tfrac{1}{2} (\partial \underline{\varphi})^{\mathrm{T}} (\partial \underline{\varphi}) + \tfrac{1}{2} \underline{\varphi}^{\mathrm{T}} \underline{\underline{m}}^2 \underline{\varphi} - y \sum_{n=1}^{2} \cos(b\, \underline{f}_n^{\mathrm{T}} \underline{\varphi}) \right), \end{aligned} \tag{5}$$

where $\underline{\varphi}$ denotes the column vector $\underline{\varphi} = (\varphi_1, \varphi_2)$ characterizing the $O(2)$ doublet, and the $\underline{f}_n$ are projectors $\underline{f}_n = (\delta_{1n}, \delta_{2n})$ whose components are given by Kronecker deltas. The Josephson-type interlayer interaction corresponds to the following dimensionful mass matrix (see, e.g., [20]):

$$\underline{\underline{m}}^2 = \begin{pmatrix} J & -J \\ -J & J \end{pmatrix}. \tag{6}$$

The mass eigenvalues are 0 and $2J$. In order to perform the mapping of the double-layer sine–Gordon model (5) onto a gas of topological excitations, we follow the scenario of [30], where the partition function of the sine–Gordon model is identically rewritten in the form of the partition function of a Coulomb gas. We should perhaps note that this mapping procedure is inspired by the treatment in chapter 31 of [26]. One expands the exponential factor of the integrand with the periodic potential in a Taylor series, expresses $\cos(b\,\underline{f}_n^{\mathrm{T}}\underline{\varphi})$ in terms of exponential functions and introduces integer-valued variables, the charges $\sigma_n = \pm 1$, that fulfil the neutrality condition. After these operations, one obtains

$$\begin{aligned} \mathcal{Z}_{\mathrm{2LSG}} &= \mathcal{N} \int [\mathcal{D}\underline{\varphi}] \exp\left(-S_{\mathrm{2LSG}}[\underline{\varphi}]\right) = \mathcal{N} \sum_{\nu=0}^{\infty} \frac{(y/2)^{2\nu}}{(2\nu)!} \sum_{n_1=1}^{2} \int \mathrm{d}^2 r_1 \ldots \sum_{n_{2\nu}=1}^{2} \int \mathrm{d}^2 r_{2\nu} \\ &\times \sum_{\substack{\sigma_1,\ldots,\sigma_\nu = \pm 1 \\ \sigma_{\nu+\gamma} = -\sigma_\gamma,\ \gamma \in \{1,\ldots\nu\}}} \int [\mathcal{D}\underline{\varphi}] \exp\left(- \int \mathrm{d}^2 r \left(\tfrac{1}{2} \underline{\varphi}^{\mathrm{T}} (-\underline{\underline{1}} \partial_\mu \partial^\mu + \underline{\underline{m}}^2) \underline{\varphi} + \mathrm{i} b \underline{\rho}^{\mathrm{T}} \underline{\varphi} \right) \right), \end{aligned} \tag{7}$$

where $\underline{\underline{1}}$ stands for the two-dimensional unit matrix which will be suppressed in the following. The charge density $\underline{\rho}(\vec{r})$, which depends on the configuration of the charges $\sigma_1, \ldots, \sigma_{2\nu}$ and on their positions $\vec{r}_1, \ldots, \vec{r}_{2\nu}$, constitutes a vector in the internal space of the fields (φ_1, φ_2) characterizing the two layers, and reads $\underline{\rho}(\vec{r}) \equiv \sum_{\alpha=1}^{2\nu} \sigma_\alpha \delta(\vec{r} - \vec{r}_\alpha) \underline{f}_{n_\alpha}$. We have thus obtained a representation in which the 2ν charges have been placed onto the two layers, with the α th charge on layer n_α. Performing the Gaussian path integral in equation (7), we obtain

$$\begin{aligned} \mathcal{Z}_{\mathrm{2LSG}} &= \mathcal{N} \sum_{\nu=0}^{\infty} \frac{(y/2)^{2\nu}}{(2\nu)!} \sum_{n_1=1}^{2} \int \mathrm{d}^2 r_1 \ldots \sum_{n_{2\nu}=1}^{2} \int \mathrm{d}^2 r_{2\nu} \\ &\times \sum_{\substack{\sigma_1,\ldots,\sigma_\nu = \pm 1 \\ \sigma_{\nu+\gamma} = -\sigma_\gamma, \gamma \in \{1,\ldots\nu\}}} \exp\left(-\frac{b^2}{2} \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \underline{\rho}^{\mathrm{T}}(-\vec{p}) (\vec{p}^{\,2} + \underline{\underline{m}}^2)^{-1} \underline{\rho}(\vec{p}) \right), \end{aligned} \tag{8}$$

where $\underline{\rho}(\vec{p}) = \sum_{\alpha=1}^{2\nu} \sigma_\alpha \exp(\mathrm{i}\vec{p} \cdot \vec{r}_\alpha) \underline{f}_{n_\alpha}$ is the Fourier transform of the $O(2)$ charge density. In momentum space, the propagator can easily be calculated by matrix inversion, and this gives

$$\mathcal{Z}_{2\mathrm{LSG}} = \mathcal{N} \sum_{\nu=0}^{\infty} \frac{(y/2)^{2\nu}}{(2\nu)!} \sum_{n_1=1}^{2} \int \mathrm{d}^2 r_1 \ldots \sum_{n_{2\nu}=1}^{2} \int \mathrm{d}^2 r_{2\nu}$$
$$\times \sum_{\substack{\sigma_1,\ldots,\sigma_\nu=\pm 1 \\ \sigma_{\nu+\gamma}=-\sigma_\gamma,\gamma\in\{1,\ldots\nu\}}} \exp\Bigg(-b^2 \sum_{\alpha=1}^{2\nu}\sum_{\gamma=1}^{2\nu} \frac{1}{2}\sigma_\alpha\sigma_\gamma \left(\delta_{1n_\alpha} \quad \delta_{2n_\alpha}\right)$$
$$\times \begin{pmatrix} A(r_{\alpha\gamma}) & B(r_{\alpha\gamma}) \\ B(r_{\alpha\gamma}) & A(r_{\alpha\gamma}) \end{pmatrix} \begin{pmatrix} \delta_{1n_\gamma} \\ \delta_{2n_\gamma} \end{pmatrix}\Bigg). \tag{9}$$

Here, the interaction potentials are ($r_{\alpha\gamma} = |\vec{r}_\alpha - \vec{r}_\gamma|$)

$$A(r_{\alpha\gamma}) = \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{\mathrm{e}^{[\mathrm{i}\vec{p}\cdot(\vec{r}_\alpha - \vec{r}_\gamma)]}(\vec{p}^2 + J)}{\vec{p}^2(\vec{p}^2 + 2J)}$$
$$= -\frac{1}{2\pi}\left(\frac{1}{2}\ln\left(\frac{r_{\alpha\gamma}}{a}\right) - \frac{1}{2}\left[K_0\left(\frac{r_{\alpha\gamma}}{\lambda_{\mathrm{eff}}}\right) - K_0\left(\frac{a}{\lambda_{\mathrm{eff}}}\right)\right]\right), \tag{10a}$$

$$B(r_{\alpha\gamma}) = \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{\mathrm{e}^{[\mathrm{i}\vec{p}\cdot(\vec{r}_\alpha - \vec{r}_\gamma)]}J}{\vec{p}^2(\vec{p}^2 + 2J)}$$
$$= -\frac{1}{2\pi}\left(\frac{1}{2}\ln\left(\frac{r_{\alpha\gamma}}{a}\right) + \frac{1}{2}\left[K_0\left(\frac{r_{\alpha\gamma}}{\lambda_{\mathrm{eff}}}\right) - K_0\left(\frac{a}{\lambda_{\mathrm{eff}}}\right)\right]\right), \tag{10b}$$

where the momentum integrals can be performed using either dimensional regularization [26] or ultraviolet (UV) cutoffs and the physically relevant, finite parts of the interaction potentials consist of massless and massive scalar propagators. In the expression for the intralayer (A) and interlayer (B) interaction potentials, a is the lattice spacing which serves as a short-distance (UV) cutoff, K_0 denotes the modified Bessel function of the second kind, and $\lambda_{\mathrm{eff}} = 1/\sqrt{2J}$ is an effective screening length. The asymptotics of the interaction potentials read as follows ($\gamma_{\mathrm{E}} = 0.57721\,56649\ldots$ is Euler's constant):

$$A(r_{\alpha\gamma} \ll \lambda_{\mathrm{eff}}) \sim -\frac{1}{2\pi}\ln\left(\frac{r_{\alpha\gamma}}{a}\right), \tag{11a}$$

$$A(r_{\alpha\gamma} \gg \lambda_{\mathrm{eff}}) \sim -\frac{1}{2\pi}\left(\frac{1}{2}\ln\left(\frac{r_{\alpha\gamma}}{\lambda_{\mathrm{eff}}}\right) + \ln\left(\frac{\lambda_{\mathrm{eff}}}{a}\right) + \frac{1}{2}\ln(2) - \frac{1}{2}\gamma_{\mathrm{E}}\right), \tag{11b}$$

$$B(r_{\alpha\gamma} \ll \lambda_{\mathrm{eff}}) \sim 0, \tag{11c}$$

$$B(r_{\alpha\gamma} \gg \lambda_{\mathrm{eff}}) \sim -\frac{1}{2\pi}\left(\frac{1}{2}\ln\left(\frac{r_{\alpha\gamma}}{\lambda_{\mathrm{eff}}}\right) + \frac{1}{2}\ln(2) - \frac{1}{2}\gamma_{\mathrm{E}}\right). \tag{11d}$$

Here, $a \ll \lambda_{\mathrm{eff}}$ is assumed. The partition function of the double-layer sine–Gordon model is thus identically rewritten in the form of a partition function for a gas of topological excitations, which we would like to call the 'layered sine–Gordon gas', and which is given by

$$\mathcal{Z}_{2\mathrm{LSG}} = \mathcal{N} \sum_{\nu=0}^{\infty} \frac{(y/2)^{2\nu}}{(2\nu)!} \sum_{n_1=1}^{2} \int \mathrm{d}^2 r_1 \ldots \sum_{n_{2\nu}=1}^{2} \int \mathrm{d}^2 r_{2\nu}$$
$$\times \sum_{\substack{\sigma_1,\ldots,\sigma_\nu=\pm 1 \\ \sigma_{\nu+\gamma}=-\sigma_\gamma,\gamma\in\{1,\ldots\nu\}}} \exp\Bigg(-b^2 \sum_{\alpha\neq\gamma}^{2\nu} \tfrac{1}{2}\sigma_\alpha\sigma_\gamma \{\delta_{n_\alpha n_\gamma} A(r_{\alpha\gamma})$$
$$+ (1 - \delta_{n_\alpha n_\gamma}) B(r_{\alpha\gamma})\}\Bigg), \tag{12}$$

where the contact terms $\alpha = \gamma$ are treated separately (the latter modification leads to a physically irrelevant renormalization of the partition function). The frequency b is inversely proportional to the temperature, $b^2 = 2\pi/(k_B T)$, and the Fourier amplitude y is related to the fugacity z by the relation $z^{2\nu}/(\nu!)^2 = (y/2)^{2\nu}/(2\nu)!$, i.e. $y = 2z\left(\frac{(\nu+1)_\nu}{\nu!}\right)^{1/(2\nu)}$, where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol.

2.3. Comparison of the layered sine–Gordon and layered vortex gas models

The partition functions (2) of the layered vortex gas and (12) of the layered sine–Gordon gas have the same structure. Therefore, the intralayer and interlayer interaction potentials can thus be compared directly. A comparison of equations (3*a*) and (10*a*) reveals that, for small distances ($r \ll \lambda_{\text{eff}}$), the intralayer potentials have the same logarithmic behaviour for both models. This is not unexpected, since in this case the vortices of a given layer are independent of the effects in the other layer. However, for large distances ($r \gg \lambda_{\text{eff}}$), the intralayer potential is logarithmic for the gas of topological excitations of the layered sine–Gordon model in contrast to the layered vortex gas, whereas the long-range intralayer potential is dominated by a linear term. The difference between the two models becomes even more significant if one compares the interlayer potentials which are different for the two models both in the short-range as well as the long-range regime (see equations (3*b*) and (10*b*)).

The significant differences of the long-range behaviour of the interlayer potentials strongly indicate different long-distance (infrared, IR) physics. We should note that the long-range behaviour of the potentials in equations (11*b*) and (11*d*) generalizes to a leading asymptotics of the form $-1/(2\pi N)\ln(r_{\alpha\gamma}/\lambda_{\text{eff}})$ for an N-layer system with the interlayer interaction given in equation (13) below. Thus, the addition of more layers does not change the qualitative behaviour of the long-range potentials (linear versus logarithmic). We conclude that there is a strong indication that these models belong to different universality classes, and that the layered sine–Gordon model is *not* suitable for describing the vortex properties of Josephson-coupled layered superconductors if a linear long-range potential between the topological defects is assumed.

3. Phase structure of the N-layer sine–Gordon model

From a conceptual point of view, it is interesting to study the critical temperature as a function of the number of coupled layers, in the framework of an appropriate generalization of the double-layer model defined in equation (5) to N layers. The discretization of the derivative term for the z-direction in the three-dimensional sine–Gordon Lagrangian results in a model of coupled 2D systems [33], which has been called the N-layer sine–Gordon model. It consists of N coupled 2D sine–Gordon models of identical 'frequency' b [22, 33], each of which corresponds to a single layer described by the scalar fields φ_i ($i = 1, 2, \ldots, N$). Its bare action reads (see equation (2) of [22])

$$S_{\text{NLSG}} = \int \mathrm{d}^2 r \left[\frac{1}{2}\sum_{i=1}^{N}(\partial_\mu\varphi_i)^2 + \sum_{i=1}^{N-1}\frac{J}{2}(\varphi_{i+1}-\varphi_i)^2 + \sum_{i=1}^{N} y_i\cos(b\varphi_i)\right]. \tag{13}$$

We have implicitly defined the mass matrix $\underline{\underline{m}}^2_N$ of the N-layer model, $\sum_{i=1}^{N-1}\frac{J}{2}(\varphi_{i+1}-\varphi_i)^2 \equiv \frac{1}{2}\underline{\varphi}^{\mathrm{T}}\underline{\underline{m}}^2_N\underline{\varphi}$. The action is invariant under a joint shift of all fields $\varphi_i \to \varphi_i + 2\pi/b$ applied to all layers $i = 1, 2, \ldots, N$, a symmetry which corresponds to a single zero-mass eigenvalue of the matrix $\underline{\underline{m}}^2_N$. Indeed, after a suitable rotation of the mass matrix [22, 33], it becomes evident that the N-layer sine–Gordon model consists of $N-1$ massive 2D sine–Gordon fields and a single massless 2D sine–Gordon field. The periodicity in the internal space spanned by the field is broken explicitly for the massive fields, and the spontaneous breaking of periodicity of the single massless mode accompanies the phase transition for small values of fugacities [20, 22].

The rotated N-layer sine–Gordon model has already been investigated by the Wegner–Houghton renormalization group method on the basis of the mass-corrected linearized scaling laws [33] and by a general perturbative treatment [22]. Both approaches predict a linear increase of the critical parameter b_c^2 with increasing number N of the layers, according to the formula $b_c^2 = 8\pi N$ (see equation (35) of [22]). Equation (12) clearly implies that $b_c^2 = 8\pi N = 2\pi/(k_B T)$, and we therefore obtain $T_c \propto 1/N$ for the N-layer model. This decrease of the transition temperature is perfectly consistent with the general properties of the model in the limit of an infinite number of layers. Namely, one can intuitively assume that the single remaining zero-mass eigenvalue cannot make a decisive contribution to the phase structure of the model in the limit $N \to \infty$, with $N - 1$ modes being massive. Indeed, in the limit of an infinite number of layers, one recovers the 3D sine–Gordon model which does not undergo any phase transition at all [30, 31, 33].

4. Summary and conclusions

The main result of this paper (see section 2) is the indirect comparison of the layered sine–Gordon model to the layered Ginzburg–Landau theory: as we have shown, both models can be mapped to different gases of topological excitations. These are the layered vortex gas for the layered Ginzburg–Landau theory (see equation (2)) and the equivalent gas of topological excitations for the layered sine–Gordon model (the 'layered sine–Gordon gas'; see equation (12)). In general, we find that if a long-range confining linear potential is required for a description of the Josephson-coupled layered high-T_c superconductors, then the system of coupled 2D sine–Gordon models is not suitable to describe the vortex properties of these materials: scalar-field propagators cannot provide linear potentials in two dimensions. For short distances, a logarithmic behaviour can of course be approximated quite well by a linear potential (see [13], $\ln(1 + r) \sim r$ for $r \ll 1$), but this observation is irrelevant for the phase structure of a system, which is determined only by the long-range interactions. In any case, we are led to the conjecture that the layered sine–Gordon and the layered vortex gas models belong to different universality classes. Using a renormalization group analysis of the generalized N-layer sine–Gordon model as described in section 3, we find that the critical temperature of the layered sine–Gordon gas reads fulfils $k_B T_c = (4N)^{-1}$. This is inconsistent with high transition temperatures for multi-layer systems and in strong disagreement with experiment [39–41]. For example, in [40], for $YBa_2Cu_3O_{7-\delta}$ the single-layer (2D) transition temperature was determined as 30.1 K, and with $N = 2$ layers, the experimental result was $T_{KTB} = 58.2$ K, suggesting $T_c \propto N$ for a small number of layers.

Let us conclude this paper with a perhaps somewhat surprising outlook. The decrease of the transition temperature with the number of layers is tied to the gradual 'disappearance' of the 'influence' of the only remaining zero-mass mode in the matrix of the Josephson-coupled layered sine–Gordon model, in comparison to the $N - 1$ massive modes, as $N \to \infty$. If we choose the mass matrix differently, e.g., $\underline{\varphi}^{\mathrm{T}} \underline{\underline{M}}^2 \underline{\varphi} = G(\sum_{n=1}^{N} a_n \varphi_n)^2$, with the (only) condition $a_n^2 = 1$, then there are $N - 1$ massless modes and only one massive mode. In that case, we find (see [42]) $T_c \propto \frac{N-1}{N}$, and this result is in agreement with the analysis presented in [43] for magnetically coupled layered superconductors. In this case, the interaction potentials corresponding to equation (11) between the topological defects have the same asymptotic behaviour as those given in [3, 43] for the magnetically coupled case. After all, a layered sine–Gordon-type field theory with a suitable interlayer interaction might prove to be useful for the description of vortex dynamics in (magnetically coupled) layered systems, but not in the expected direction, which would have been the Josephson-coupled case.

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